On 3×3 strongly clean upper triangular matrices

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Abstract

In this paper, we construct all classes of 3×3 strongly clean upper triangular matrices over $\mathbb{Z}$. Moreover, necessary and sufficient conditions under which a 3×3 upper triangular matrix over $\mathbb{Z}$ is strongly clean are given.

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1 Introduction

An element in a ring R is stongly clean provided that it is the sum of an idempotent and a unit that commute with each other. This notion was firstly introduced by Nicholson in [6]. Let $T_3(R)$ be the ring consisting of 3×3 upper triangular matrices over R and $U(T_3(R))$ be the subset of $T_3(R)$ consisting of invertible matrices.

In [3], Chen studied 2×2 strongly clean matrices and gave several necessary and sufficient conditions under which a 2×2 matrix over an integral domain is strongly clean. He showed that strong cleanness over integral domains can be characterized by quadratic and Diophantine equations.

The main purpose of this note is to determine the strong cleanness of the matrices of the forms

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

where $a, b, c, d, e, f \in \mathbb{Z}$. We give the necessary and sufficient conditions under which such 3×3 upper triangular matrices are strongly clean.

2 Strongly clean matrices

For a discussion about 3×3 strongly clean upper triangular matrices, at first the general forms of the 3×3 idempotent upper triangular matrices are needed. Let $E \in T_3(\mathbb{Z})$ be an idempotent matrix ($E^2 = E$). Then $(\det(E))^2 = \det(E^2) = \det(E)$. Thus, $\det(E) = 0$ or 1. Now we give the following proposition:

Proposition 2.1. *Let $\mathbb{Z}$ be the ring of integers. Then $E = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \in T_3(\mathbb{Z})$ is a non-zero idempotent matrix if and only if it is one of the following forms:*

$$I, \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & ab \\ 0 & 1 & b \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & a & -ab \\ 0 & 0 & b \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

that $a, b \in \mathbb{Z}$.

Proof. According to the above description, and also noting that the determinant of an upper triangular matrix is the product of the main diagonal elements, the proof is obvious. For example suppose that $E = \begin{pmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ is an idempotent matrix. Then, we must have

$$\begin{pmatrix} 1 & a & b+ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left(\begin{pmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \right)^2 = E^2 = E = \begin{pmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

So, we conclude that $c = 0$, and as a result, $E = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is an idempotent matrix. □

Lemma 2.2. *Let $A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$ be a matrix on the ring $T_3(\mathbb{Z})$. Then, the inverse of integer matrix A is again an integer matrix if and only if the determinant of A is exactly 1 or -1.*

Proof. Let A^{-1} be the inverse of A. Then,

$$A^{-1} = (\det(A))^{-1}\mathrm{adj(A)} = (adf)^{-1} \begin{pmatrix} df & \star & \star \\ 0 & af & \star \\ 0 & 0 & ad \end{pmatrix} \Rightarrow a^{-1}, d^{-1}, f^{-1} \in \mathbb{Z}.$$

Hence, $a, d, f = \pm 1$ and so we obtain $\det(A) = \pm 1$.

Conversely, if $\det(A) = \pm 1$, then $A^{-1} = (\det(A))^{-1}\mathrm{adj(A)} = \pm\mathrm{adj(A)}$ is an integer matrix. $\square$

Proposition 2.3. *Let* $A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \in T_3(\mathbb{Z})$*, and let* $A = E + U$ *such that* E *is an idempotent upper triangular matrix, and* U *is an invertible upper triangular matrix. Then,*

(1) If $E = 0$*, then* $A - E$ *is unit matrix if and only if* A *is unit matrix.*

(2) If $E = \begin{pmatrix} 1 & g & h \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $g, h \in \mathbb{Z}$*, then* $A - E$ *is unit matrix if and only if* $a = 0$ *or* 2, $d, f = \pm 1$.

(3) If $E = \begin{pmatrix} 0 & g & gh \\ 0 & 1 & h \\ 0 & 0 & 0 \end{pmatrix}$ *and* $g, h \in \mathbb{Z}$*, then* $A - E$ *is unit matrix if and only if* $d = 0$ *or* 2, $a, f = \pm 1$.

(4) If $E = \begin{pmatrix} 0 & 0 & g \\ 0 & 0 & h \\ 0 & 0 & 1 \end{pmatrix}$ *and* $g, h \in \mathbb{Z}$*, then* $A - E$ *is unit matrix if and only if* $f = 0$ *or* 2, $a, d = \pm 1$.

(5) If $E = \begin{pmatrix} 1 & 0 & g \\ 0 & 1 & h \\ 0 & 0 & 0 \end{pmatrix}$ *and* $g, h \in \mathbb{Z}$*, then* $A - E$ *is unit matrix if and only if* $a, d = 0$ *or* 2, $f = \pm 1$.

(6) If $E = \begin{pmatrix} 1 & g & h \\ 0 & 0 & k \\ 0 & 0 & 1 \end{pmatrix}$ *and* $g, h, k \in \mathbb{Z}$*, and* $gk + h = 0$*, then* $A - E$ *is unit matrix if and only if* $a, f = 0$ *or* 2, $d = \pm 1$.

(7) If $E = \begin{pmatrix} 0 & g & h \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *and* $g, h \in \mathbb{Z}$*, then* $A - E$ *is unit matrix if and only if* $d, f = 0$ *or* 2, $a = \pm 1$.

Proof. The verification is straightforward. For example we prove (6). Let $A - E$ be unit. Then, by Lemma 2.2

$$\det(A - E) = \begin{vmatrix} a-1 & b-g & c-h \\ 0 & d & e-k \\ 0 & 0 & f-1 \end{vmatrix} = (a-1)d(f-1) = \pm 1.$$

So $a, f = 0$ or 2, $d = \pm 1$.

Conversely, let $(a-1)d(f-1) = \pm 1$. Then, $\det(A-E) = \pm 1$. Hence, according to Lemma 2.2, $A - E$ is unit. $\square$

Let $A \in T_3(\mathbb{Z})$. We say that A is strongly 0-clean (strongly 1-clean) in the case that there exists an idempotent $E \in T_3(\mathbb{Z})$ such that $A - E$ is unit, $AE = EA$, and $\det(E) = 0$ ($\det(E) = 1$).

Theorem 2.4. *Let* $A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \in T_3(\mathbb{Z})$*. Then, A is strongly 1-clean if and only if* $\mathrm{tr}(A) + \det(A) - ad - af - df = 0$ *or* 2.

Proof. By Proposition 2.1, the identity matrix is the only integer 3×3 idempotent matrix with determinant equal to one. Then,

$$\begin{aligned} A \text{ is strongly } 1-\text{clean } &\Leftrightarrow \text{ there exists } U \in U(T_3(\mathbb{Z})) \text{ such that } A = I_3 + U \\ &\Leftrightarrow U = \begin{pmatrix} a-1 & b & c \\ 0 & d-1 & e \\ 0 & 0 & f-1 \end{pmatrix} \in U(T_3(\mathbb{Z})) \\ &\Leftrightarrow \det(U) = \pm 1 \\ &\Leftrightarrow adf - ad - af + a - df + d + f - 1 = \pm 1 \\ &\Leftrightarrow \mathrm{tr}(A) + \det(A) - ad - af - df = 0 \text{ or } 2 \end{aligned}$$

$\square$

Example 1. $A = \begin{pmatrix} 2 & 3 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(\mathbb{Z})$ is strongly 1-clean, because

$$\mathrm{tr}(A) + \det(A) - ad - af - df = 4 + 0 - 0 - 4 - 0 = 0.$$

Moreover, we have

$$A = \begin{pmatrix} 2 & 3 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{Idempotent}} + \underbrace{\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{pmatrix}}_{\text{Unit}}.$$

Theorem 2.5. *Let $\mathbb{Z}$ be the ring of integers, and let* $A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \in T_3(\mathbb{Z})$*. Then, A is strongly 0-clean if and only if*

(1) A is invertible, or

(2) $\det(A) - df = \pm 1$, *and the system of equations*

$$\begin{cases} (a-d)x - b = 0 \\ -ex + (a-f)y - c = 0 \end{cases}$$

is solvable, or

(3) $\det(A) - af = \pm 1$, *and the system of equations*

$$\begin{cases} (d-a)x - b = 0 \\ (d-f)y - e = 0 \\ ex - by + (f-a)xy = 0 \end{cases}$$

is solvable, or

(4) $\det(A) - ad = \pm 1$, *and the system of equations*

$$\begin{cases} (f-a)x - by - c = 0 \\ (f-d)y - e = 0 \end{cases}$$

is solvable, or

(5) $\det(A) - f(a+d-1) = \pm 1$, *and the system of equations*

$$\begin{cases} (a-f)x + by - c = 0 \\ (d-f)y - e = 0 \end{cases}$$

is solvable, or

(6) $\det(A) - d(a+f-1) = \pm 1$, *and the system of equations*

$$\begin{cases} (a-d)x - b = 0 \\ ex + (f-a)y - bz = 0, \ (y = -xz) \\ (f-d)z - e = 0 \end{cases}$$

is solvable, or

(7) $\det(A) - a(d+f-1) = \pm 1$, *and the system of equations*

$$\begin{cases} (d-a)x - b = 0 \\ ex + (f-a)y - c = 0 \end{cases}$$

is solvable.

Proof. By Proposition 2.3, the proof is clear. $\square$

Example 2. Suppose that $A = \begin{pmatrix} 1 & 2 & -5 \\ 0 & 2 & 9 \\ 0 & 0 & -1 \end{pmatrix}$. Then, A is strongly 0-clean, because

$$A = \begin{pmatrix} 1 & 2 & -5 \\ 0 & 2 & 9 \\ 0 & 0 & -1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}}_{Idempotent} + \underbrace{\begin{pmatrix} 1 & 0 & -11 \\ 0 & 1 & 6 \\ 0 & 0 & -1 \end{pmatrix}}_{Unit}.$$

But, A in not strongly 1-clean, because

$$\operatorname{tr}(A) + \det(A) - ad - af - df = 2 + (-2) - 2 - (-1) - (-2) = 1.$$

3 Strongly π-regular and Strongly J_n-clean

In this section, we begin with the definition of strongly π-regular.

Definition 3.1. Let R be a ring and A a 3×3 upper triangular matrix over R. Then, A is strongly π-regular if $A^n T_3(R) = A^{n+1} T_3(R)$ for som integer $n \geqslant 1$.

Azumaya's Lemma. [1] If $a \in R$ is strongly π-regular and $a^n R = a^{n+1} R$ for some integer $n \geqslant 1$, then a^n is strongly regular and there exists $b \in R$ such that $ab = ba$ and $a^n = a^{n+1} b$ ($c \in R$ is strongly regular if $c = ue = eu$ for some $u \in U(R)$ and some e such that $e^2 = e \in R$).

Proposition 3.2. *Let $\mathbb{Z}$ be the ring of integers, and $A \in T_3(\mathbb{Z})$. Then, the following are equivalent:*

(1) A is strongly π-regular.

(2) There exists $n \in \mathbb{N}$ such that $A^n = FW = WF$ where $F^2 = F$, $W \in U(T_3(\mathbb{Z}))$ and A, F, W mutually commute.

(3) A^m is strongly regular for some m in $\mathbb{N}$.

Proof. (1) $\Rightarrow$ (2). If A is strongly π-regular, Azumaya's lemma provides $B \in T_3(\mathbb{Z})$ and $1 \leqslant n \in \mathbb{Z}$ such that $AB = BA$ and $A^n = A^{n+1}B$. Thus, $A^n = A^n(AB) = A^{n+1}B(AB) = A^{n+2}B^2$. If we iterate this, we get $A^n = A^{n+k}B^k$ for each $k \geqslant 1$. In particular, $A^n = A^{2n}B^n = A^n B^n A^n$. Thus, $F = A^n B^n = B^n A^n$ is an idempotent, $AF = FA$ and $A^n F = A^n$. If $C = B^n F$ then $C \in F T_3(\mathbb{Z}) F$ and $A^n C = C A^n = F$. Hence $W = A^n + (I - F)$ is a unit with $W^{-1} = C + (I - F)$, and since $FW = WF = A^n F = A^n$, we are done.

(2) $\Rightarrow$ (3). Let $A^n = WF = FW$. Then,

$$A^n = FW \Rightarrow A^n W^{-1} = F \Rightarrow A^n W^{-1} A^n = F A^n \Rightarrow A^n W^{-1} A^n = A^n$$

and $AW^{-1} = W^{-1}A$.

(3) $\Rightarrow$ (1). If A^m is strongly regular where $m \geqslant 1$, then $A^m T_3(\mathbb{Z}) = A^{2m} T_3(\mathbb{Z})$ so $T_3(\mathbb{Z})A^m = T_3(\mathbb{Z})A^{2m}$ by the above conditions, which implies (1). □

Definition 3.3. Let R be a ring. Then, an element $A \in T_3(R)$ is strongly J_n-clean provided that there exists an idempotent $E \in T_3(R)$ such that $A - E \in U(T_3(R))$, $AE = EA$ and $(EA)^n \in J(T_3(R))$ ($J(T_3(R))$ is Jacobson radical of $T_3(R)$).

Remark 1. [7] Let K be a division ring, and R be the ring of upper triangular 3×3 matrices with entries in K. Let L be the subset of R consisting of matrices with zeros on the main diagonal. Then, $L = J(R)$.

In the general case, $A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \in T_3(K)$ is strongly J_n-clean, if $A = E+U$ is a representation of a strongly clean and $(EA)^n \in J(T_3(K))$. Then,

$$(EA)^n = EA^n = E\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}^n = E\begin{pmatrix} a^n & \star & \star \\ 0 & d^n & \star \\ 0 & 0 & f^n \end{pmatrix} \in J(T_3(K)).$$

Now by Proposition 2.1, we have

Case 1. If $E = \begin{pmatrix} 1 & g & h \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then

$$E\begin{pmatrix} a^n & \star & \star \\ 0 & d^n & \star \\ 0 & 0 & f^n \end{pmatrix} = \begin{pmatrix} 1 & g & h \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a^n & \star & \star \\ 0 & d^n & \star \\ 0 & 0 & f^n \end{pmatrix} = \begin{pmatrix} a^n & \star & \star \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in J(T_3(K)).$$

Therefore, A is strongly J_n-clean, if $a = 0$.

Case 2. If $E = \begin{pmatrix} 0 & g & gh \\ 0 & 1 & h \\ 0 & 0 & 0 \end{pmatrix}$, then

$$E\begin{pmatrix} a^n & \star & \star \\ 0 & d^n & \star \\ 0 & 0 & f^n \end{pmatrix} = \begin{pmatrix} 0 & g & gh \\ 0 & 1 & h \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a^n & \star & \star \\ 0 & d^n & \star \\ 0 & 0 & f^n \end{pmatrix} = \begin{pmatrix} 0 & \star & \star \\ 0 & d^n & \star \\ 0 & 0 & 0 \end{pmatrix} \in J(T_3(K)).$$

Therefore, A is strongly J_n-clean, if $d = 0$.

Case 3. If $E = \begin{pmatrix} 0 & 0 & g \\ 0 & 0 & h \\ 0 & 0 & 1 \end{pmatrix}$, then

$$E\begin{pmatrix} a^n & \star & \star \\ 0 & d^n & \star \\ 0 & 0 & f^n \end{pmatrix} = \begin{pmatrix} 0 & 0 & g \\ 0 & 0 & h \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a^n & \star & \star \\ 0 & d^n & \star \\ 0 & 0 & f^n \end{pmatrix} = \begin{pmatrix} 0 & 0 & \star \\ 0 & 0 & \star \\ 0 & 0 & f^n \end{pmatrix} \in J(T_3(K)).$$

Therefore, A is strongly J_n-clean, if $f=0$.

Case 4. If $E=\begin{pmatrix}1&0&g\\0&1&h\\0&0&0\end{pmatrix}$, then

$$E\begin{pmatrix}a^n&\star&\star\\0&d^n&\star\\0&0&f^n\end{pmatrix}=\begin{pmatrix}1&0&g\\0&1&h\\0&0&0\end{pmatrix}\begin{pmatrix}a^n&\star&\star\\0&d^n&\star\\0&0&f^n\end{pmatrix}=\begin{pmatrix}a^n&\star&\star\\0&d^n&\star\\0&0&0\end{pmatrix}\in J(T_3(K)).$$

Therefore, A is strongly J_n-clean, if $a=d=0$.

Case 5. If $E=\begin{pmatrix}1&g&h\\0&0&k\\0&0&1\end{pmatrix}$, then

$$E\begin{pmatrix}a^n&\star&\star\\0&d^n&\star\\0&0&f^n\end{pmatrix}=\begin{pmatrix}1&g&h\\0&0&k\\0&0&1\end{pmatrix}\begin{pmatrix}a^n&\star&\star\\0&d^n&\star\\0&0&f^n\end{pmatrix}=\begin{pmatrix}a^n&\star&\star\\0&0&\star\\0&0&f^n\end{pmatrix}\in J(T_3(K)).$$

Therefore, A is strongly J_n-clean, if $a=f=0$.

Case 6. If $E=\begin{pmatrix}0&g&h\\0&1&0\\0&0&1\end{pmatrix}$, then

$$E\begin{pmatrix}a^n&\star&\star\\0&d^n&\star\\0&0&f^n\end{pmatrix}=\begin{pmatrix}0&g&h\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}a^n&\star&\star\\0&d^n&\star\\0&0&f^n\end{pmatrix}=\begin{pmatrix}0&\star&\star\\0&d^n&\star\\0&0&f^n\end{pmatrix}\in J(T_3(K)).$$

Therefore, A is strongly J_n-clean, if $d=f=0$.

Conclusion. All classes of 3×3 strongly clean upper triangular matrices over $\mathbb{Z}$ are constructed and also necessary and sufficient conditions under which a 3×3 upper triangular matrix over $\mathbb{Z}$ is strongly clean are provided. Moreover strongly π-regular matrices have been explained and all 3×3 strongly J_n-clean upper triangular matrices have been determined.

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